Non-linear interactions in a viscous heat-conducting compressible gas

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SUMMARY

The linearized equations of motion show that in a viscous heat-conducting compressible medium three modes of fluctuations exist, each one of which is a familiar type of disturbance. The vorticity mode occurs in an incompressible turbulent flow, the entropy mode is familiar as temperature fluctuations in low speed turbulent heat transfer problems, and the sound mode is the subject of conventional acoustics. A consistent higher order perturbation theory is presented with the only restrictions being that the Prandtl number is $\frac{3}{4}$ and the viscosity and heat conductivity are monotonic functions of the temperature alone. The theory is based on expansion of the disturbance fields in powers of an amplitude parameter α. The non-linearity of the full Navier–Stokes equations can be interpreted as interaction between the three basic modes; in order to help physical insight the interactions are classed as ' mass-like ', ' force-like ', and ' heat-like ' effects.

Besides the amplitude parameter α there is another subsidiary non-dimensional parameter ϵ which indicates the relative importance of viscosity and heat conduction effects as compared to the inertial effects, ϵ is proportional to the ratio of the molecular mean free path and the characteristic length of the flow pattern (Knudsen number). The main contribution of the paper is the outline of a consistent successive approximation for an arbitrary order in α and the presentation of explicit formulae for the second order (bilateral) interactions.

A special case of rather general significance is treated in more detail. This is when all three basic modes have intensities and length scales of the same orders of magnitude and in addition to α the parameter ϵ is also small; the second-order interactions are then relatively few and easily identifiable and are shown in table 1.

The present analysis also sheds some light on the ' zero order ' approximation which treats the vorticity and entropy disturbances as a ' frozen pattern ' and the sound field as propagating non-dissipative waves. The interpretation of hot-wire measurements relies heavily on these simplified models and the present paper lends some support to these current hot-wire practices.

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1. Introduction

In fluid mechanics the non-linearity of the governing equations manifests itself in several ways. Some of the non-linear effects are described by various special names, for instance, the continuous generation of vorticity in the phenomenon called turbulence, the gradual steepening of the slope of compression waves, the generation of aerodynamic sound, acoustic streaming, scattering of sound waves by turbulence, etc. The question arises of whether all types of non-linear phenomena governed by the compressible Navier–Stokes equations have been uncovered or whether there are still others that have escaped attention. This can be answered only by a systematic study of the equations.

There is another important question; by varying some overall parameter (e.g. the Mach number of the mean flow, the length scale of the fluctuation, etc.) the relative importance of the various non-linear effects may vary. A typical problem is the change in the nature of turbulence as the Mach number of the primary flow increases. In actual experimentation the measured quantities must be interpreted within a theoretical framework even if it serves only to guide our thinking. The measurement of fluctuating quantities in a turbulent flow has been extended to high speed flows (Kovásznay 1953) and the present approach was strongly motivated by the desire to generalize the crude but simple concepts that were obtained from the linearized problem.

Both at low speeds and at high speeds, the hot-wire measurements are carried out keeping the probe at a fixed location and the records are obtained as fluctuations in time and all statistical averaging is performed in the time domain. These time records at low speeds are often regarded as approximate instantaneous space traverses by virtue of Taylor's hypothesis that the flow pattern is carried with the main flow and changes only relatively slowly. In high speed flow the simple linearized theory of the compressible case indicated that there are three fluctuation modes and for two of them (vorticity and entropy modes) this simple motion of 'frozen pattern' can be immediately extended. We may call these 'parabolic' modes as they obey parabolic differential equations. The third mode, the pressure or sound mode, obeys a hyperbolic differential equation and can be interpreted as a system of acoustic waves travelling with respect to the main flow.

The linearized theory does not indicate any interaction among the modes as long as the domain of interest is far from solid boundaries and it may be valid as long as the fluctuations are weak. When the intensity of fluctuation ceases to be small, the flow can still be conveniently interpreted as built up by superposition of the three basic modes of fluctuation. However, in addition to those originally accounted for, there are also fluctuations produced by the interaction of the basic modes. A systematic study of these interactions is expected to cover all the possible non-linear phenomena.

The analysis presented here does not attempt to 'solve' compressible heat conducting viscous flow problems in the sense of an initial value or boundary value problem. It attempts rather to serve as a guide to assess all the non-linear interactions in a consistent framework in which experimental information (of the usual incomplete nature) can be understood.

2. Fundamental equations

The fundamental system of differential equations governing the motion of a viscous, heat-conducting, compressible medium consists of the continuity, momentum and energy equations. If p, ρ, T, E, $\mathbf{v}$, μ and κ denote respectively the pressure, density, temperature, entropy, velocity, coefficients of viscosity and heat conduction, the fundamental equations assume the following form:

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \mathbf{v} = m, \tag{2.1}$$

$$\rho \frac{D\mathbf{v}}{Dt} = -\operatorname{grad} p - \tfrac{2}{3} \operatorname{grad}(\mu \operatorname{div} \mathbf{v}) + \operatorname{div}[\mu \operatorname{def} \mathbf{v}] + \rho \mathbf{f}, \tag{2.2}$$

$$\frac{p}{R} \frac{DE}{Dt} = \operatorname{div}[\kappa \operatorname{grad} T] + \mu[\tfrac{1}{2}(\operatorname{def} \mathbf{v}):(\operatorname{def} \mathbf{v}) - \tfrac{2}{3}(\operatorname{div} \mathbf{v})^2] + Q, \tag{2.3}$$

where D/Dt is Stokes material derivative and $\operatorname{def} \mathbf{v}$ is the rate of deformation tensor. (If the components of the velocity $\mathbf{v}$, referred to axes $Ox_1x_2x_3$, are v_1, v_2, v_3, then $\operatorname{def} \mathbf{v}$ is the second order tensor whose ij-th component is $(\partial v_i/\partial x_j + \partial v_j/\partial x_i)$ and $(\operatorname{def} \mathbf{v}):(\operatorname{def} \mathbf{v})$ denotes the double tensor product $\sum\sum (\partial v_i/\partial x_j + \partial v_j/\partial x_i)^2$.) In the equations, m denotes the rate of mass production (or injection) per unit volume, $\mathbf{f}$ denotes the body force per unit mass and Q denotes the rate of heat addition per unit volume.

The coefficients of viscosity μ and heat conduction κ are assumed to be known monotonic functions of the temperature, i.e. $\mu = \mu(T)$ and $\kappa = \kappa(T)$.

For a perfect gas, the variables p, ρ, T and E are related by the equations of state

$$p = \rho R T \tag{2.4}$$

or

$$E - E_r = C_v \log\left(\frac{p}{p_r}\right)\left(\frac{\rho_r}{\rho}\right)^{\gamma}, \tag{2.5}$$

where $\gamma = C_p/C_v$, C_p being the specific heat at constant pressure and C_v that at constant volume. The subscript r designates some standard reference state.

If m, $\mathbf{f}$, Q are known functions of x_i and t, the above five equations, together with the boundary conditions, are expected to be sufficient to determine the five unknowns p, ρ, T, E and $\mathbf{v}$. We shall be ultimately interested in the case $m = 0$, $\mathbf{f} = 0$, and $Q = 0$, but in the scheme of successive approximations the effect of the lower order terms on the higher ones will be expressed most conveniently through these apparent 'external' forcing functions.

Let us consider a four-dimensional space-time domain G, the size of which is determined by several considerations discussed in the ensuing paragraphs. Let

$$V = \iint d\boldsymbol{\xi} d\tau$$

denote the four-dimensional volume of G where $d\boldsymbol{\xi}$ stands for the volume element $d\xi_1 d\xi_2 d\xi_3$ and $d\tau$ for the element of time. We define the mean values of the pressure and entropy in G by

$$p_0 = \frac{1}{V}\iint_G p(\boldsymbol{\xi}, \tau)\, d\boldsymbol{\xi} d\tau, \tag{2.6}$$

$$E_0 = \frac{1}{V}\iint_G E(\boldsymbol{\xi}, \tau)\, d\boldsymbol{\xi} d\tau, \tag{2.7}$$

and define ρ_0, T_0, μ_0, κ_0 by the equations*

$$p_0 = \rho_0 R T_0, \tag{2.8}$$

$$E_0 = E_r + C_v \log\left(\frac{p_0}{p_r}\right)\left(\frac{\rho_r}{\rho_0}\right)^{\gamma}, \tag{2.9}$$

$$\mu_0 = \mu(T_0), \qquad \kappa_0 = \kappa(T_0). \tag{2.10}$$

Finally, the coordinate system is chosen so that

$$\mathbf{v}_0 = \frac{1}{V}\iint_G \mathbf{v}(\boldsymbol{\xi}, \tau)\, d\xi d\tau = 0, \tag{2.11}$$

that is, the coordinate system moves with the mean velocity of the fluid.

It should be noted that the mean values p_0, E_0, etc., depend upon the size of the space-time domain G. Consequently the fluctuations $p-p_0$, $E-E_0$, etc., as well as the 'intensity' of the fluctuations, $(p-p_0)/p_0$, $(E-E_0)/E_0$, etc., also depend on the size of G. If we assume that p, E and all other flow variables are continuous functions of the space and time coordinates, then the 'intensity' of the fluctuations in G can be made arbitrarily small by limiting our consideration to a sufficiently small space-time domain G. Let α be a non-dimensional parameter characterizing the intensity of the disturbance. For example, we may take α as the maximum of $|p-p_0|/|p_0|$, $|v/a_0|$ etc., where a_0 is the velocity of sound of the mean state.

We now assume that in a sufficiently small space-time domain G, the pressure, density, temperature, entropy, p, ρ, T etc., can be expanded as power series in α. This is the first condition which limits the extent of the domain G. The actual size of G is determined by the values of the spatial and temporal gradients of the fluctuations relative to our coordinate system.

* Note that ρ_0, T_0, etc. are not defined in the same manner as p_0 and E_0 mainly to simplify the algebra. The ultimate physical conclusions are not affected by the manner in which ρ_0, etc. are defined.

For low speed turbulence Lin's (1953) analysis indicates that the ratio of the temporal gradients to the spatial gradients may be small (compared with the mean velocity), in fact of the order of the velocity fluctuations themselves. The domain G would then be elongated along the time axis as can also be suspected from equations (6.5). For very small fluctuations, such as during final period of decay of grid turbulence, G may be quite sizeable.

We write

$$p = p_0 + p^{(1)} + p^{(2)} + \ldots, \tag{2.12 a}$$

$$\rho = \rho_0 + \rho^{(1)} + \rho^{(2)} + \ldots, \tag{2.12 b}$$

where $p^{(1)}$, $\rho^{(1)}$ are of order α, and $p^{(n)}$, $\rho^{(n)}$ are of order α^n.

Substituting (2.12) into the fundamental system of equations (2.1) to (2.5) inclusive and collecting terms of the same order in α, we find

$$\frac{\partial \rho^{(n)}}{\partial t} + \rho_0 \nabla . \mathbf{v}^{(n)} = F_1^{(n)}, \tag{2.13 a}$$

$$\rho_0 \frac{\partial \mathbf{v}^{(n)}}{\partial t} + \nabla p^{(n)} - \mu_0 \nabla^2 \mathbf{v}^{(n)} - \tfrac{1}{3}\mu_0 \nabla(\nabla . \mathbf{v}^{(n)}) = \mathbf{F}_2^{(n)}, \tag{2.13 b}$$

$$\frac{p_0}{R} \frac{\partial E^{(n)}}{\partial t} - \kappa_0 \nabla^2 T^{(n)} = F_3^{(n)}, \tag{2.13 c}$$

$$\frac{p^{(n)}}{p_0} - \frac{\rho^{(n)}}{\rho_0} - \frac{T^{(n)}}{T_0} = F_4^{(n)}, \tag{2.13 d}$$

$$\frac{E^{(n)}}{R} - \frac{\gamma}{\gamma - 1} \frac{T^{(n)}}{T_0} + \frac{p^{(n)}}{p_0} = F_5^{(n)}. \tag{2.13 e}$$

The functions $F_i^{(n)}$ for $n > 1$ represent the non-linearity of the original equations. For $n = 1$,

$$F_1^{(1)} = m, \qquad \mathbf{F}_2^{(1)} = \rho\mathbf{f}, \qquad F_3^{(1)} = Q, \qquad F_4^{(1)} = 0, \qquad F_5^{(1)} = 0.$$

For $n > 1$, the $F^{(n)}$ depend only on the $p^{(k)}$, $\rho^{(k)}$, etc., where $k \leq n-1$. For example, $F_1^{(2)} = -\nabla . [\rho^{(1)}\mathbf{v}^{(1)}]$. The principal advantage of the scheme presented here is that the differential equations are the same for all values of n, only the 'inhomogeneous terms' are different, and in the scheme of successive approximation these are computed from the solutions of the lower order equations.

It is convenient now to eliminate two of the four thermodynamic variables and introduce non-dimensional quantities for the remaining two. We write

$$P^{(n)} = \frac{p^{(n)}}{\gamma p_0}, \tag{2.14}$$

$$S^{(n)} = \frac{E^{(n)}}{C_p}. \tag{2.15}$$

The speed of sound of the undisturbed medium is $a_0^2 = \gamma p_0/\rho_0$. We assume further that the Prandtl number $(\mu C_p/\kappa) = \frac{3}{4}$, the sole purpose of this being to simplify the algebra.

The governing equations now become

$$\nabla . \mathbf{v}^{(n)} + \frac{\partial P^{(n)}}{\partial t} - \frac{\partial S^{(n)}}{\partial t} = \frac{m^{(n)}}{\rho_0}, \tag{2.16 a}$$

$$\frac{\partial \mathbf{v}^{(n)}}{\partial t} + a_0^2 \nabla P^{(n)} - \nu_0 \nabla^2 \mathbf{v}^{(n)} - \tfrac{1}{3}\nu_0 \nabla(\nabla . \mathbf{v}^{(n)}) = \mathbf{f}^{(n)}, \tag{2.16 b}$$

$$\frac{\partial S^{(n)}}{\partial t} - \tfrac{4}{3}\nu_0 \nabla^2 S^{(n)} - \tfrac{4}{3}(\gamma-1)\nu_0 \nabla^2 P^{(n)} = \frac{\underset{\sim}{Q}^{(n)}}{\rho_0 C_p T_0}, \tag{2.16 c}$$

where $\nu_0 = \mu_0/\rho_0$ denotes the kinematic viscosity of the undisturbed medium and

$$m^{(1)} = m, \qquad \mathbf{f}^{(1)} = \mathbf{f}, \qquad \underset{\sim}{Q}^{(1)} = \underset{\sim}{Q}.$$

The higher order $m^{(n)}$, $\mathbf{f}^{(n)}$, $\underset{\sim}{Q}^{(n)}$ are functions of the lower order solutions For the particular case $n = 2$, we find

$$\frac{m^{(2)}}{\rho_0} = -\nabla[(P^{(1)} - S^{(1)})\mathbf{v}^{(1)}] + \frac{1}{2}\frac{\partial}{\partial t}[(\gamma-1)(P^{(1)})^2 + (S^{(1)})^2 + P^{(1)}S^{(1)}], \tag{2.17 a}$$

$$\mathbf{f}^{(2)} = -(P^{(1)} - S^{(1)})\frac{\partial \mathbf{v}^{(1)}}{\partial t} - (\mathbf{v}^{(1)} . \nabla)\mathbf{v}^{(1)} - \tfrac{2}{3}\nabla\left[\frac{\mu^{(1)}}{\rho_0}(\nabla . \mathbf{v}^{(1)})\right] + \\ + \nabla . \left[\frac{\mu^{(1)}}{\rho_0} \operatorname{def} \mathbf{v}^{(1)}\right], \tag{2.17 b}$$

$$\frac{\underset{\sim}{Q}^{(2)}}{\rho_0 C_p T_0} = -\gamma P^{(1)} \frac{\partial S^{(1)}}{\partial t} - (\mathbf{v}^{(1)} . \nabla)S^{(1)} + \tfrac{4}{3}\nabla . \left[\frac{\mu^{(1)}}{\rho_0}\nabla\{S^{(1)} + (\gamma-1)P^{(1)}\}\right] + \\ + \frac{\nu_0}{C_p T_0}[\tfrac{1}{2}(\operatorname{def}\mathbf{v}^{(1)}) : (\operatorname{def}\mathbf{v}^{(1)}) - \tfrac{2}{3}(\nabla . \mathbf{v}^{(1)})^2] - \\ - \frac{2}{3}\frac{\gamma-1}{\gamma}\nu_0 \nabla^2\left[\gamma(P^{(1)})^2 - 2\gamma P^{(1)}S^{(1)} - \frac{\gamma}{\gamma-1}(S^{(1)})^2\right]. \tag{2.17 c}$$

In these equations, $\mu^{(1)}/\rho_0$ is a function of $P^{(1)}$ and $S^{(1)}$, for, by Taylor's expansion,

$$\mu(T) = \mu_0 + \left(\frac{d\mu}{dT}\right)_{T=T_0}(T - T_0) + \frac{1}{2!}\left(\frac{d^2\mu}{dT^2}\right)_{T=T_0}(T-T_0)^2 + \dots \\ = \mu_0 + T^{(1)}\left(\frac{d\mu}{dT}\right)_{T=T_0} + \left[T^{(2)}\left(\frac{d\mu}{dT}\right)_{T=T_0} + \frac{(T^{(1)})^2}{2}\left(\frac{d^2\mu}{dT^2}\right)_{T=T_0}\right] + \dots,$$

so that $\mu^{(1)} = T^{(1)}(d\mu/dT)_{T=T_0}$, $\mu^{(2)} = T^{(2)}(d\mu/dT)_{T=T_0} + \tfrac{1}{2}(T^{(1)})^2(d^2\mu/dT^2)_{T=T_0}$, etc. If we define $\mu_0' = (d\mu/dT)_{T=T_0}$, then

$$\frac{\mu^{(1)}}{\rho_0} = \frac{\mu_0'}{\rho_0}T^{(1)} = \frac{\mu_0'}{\mu_0}\nu_0 T_0[S^{(1)} + (\gamma-1)P^{(1)}].$$

A considerable simplification follows from the fact that the systems of equations (2.16) are identical in form for all values of n. The 'source functions' $m^{(1)}$, $\mathbf{f}^{(1)}$, $\underset{\sim}{Q}^{(1)}$ correspond to the actual rate of fluid injection, body force and rate of heat addition, if any, while $m^{(2)}$, $\mathbf{f}^{(2)}$, $\underset{\sim}{Q}^{(2)}$ are calculated from the solution of the first-order equations and represent only *apparent* rates of mass, momentum, and heat injection. This concept allows us to study the behaviour of the fluid motion governed by (2.1) to (2.5) in two steps. Firstly, we treat the system (2.16) as if $m^{(n)}$, $\mathbf{f}^{(n)}$, $\underset{\sim}{Q}^{(n)}$ were known and secondly, examine the structure of the quantities $m^{(n)}$, $\mathbf{f}^{(n)}$, $\underset{\sim}{Q}^{(n)}$.

3. The three basic modes of fluctuations and their production

In a sufficiently small space-time domain G the intensity of the fluctuations in the flow field can be made arbitrarily small and terms of order α^2 and higher powers of α in (2.12) may be neglected. We shall take $n = 1$ in equation (2.16), temporarily suppress the superscript (1), and obtain for the first-order fluctuations

$$\nabla.\mathbf{v} + \frac{\partial P}{\partial t} - \frac{\partial S}{\partial t} = \frac{m}{\rho_0}, \tag{3.1 a}$$

$$\frac{\partial \mathbf{v}}{\partial t} + a_0^2 \nabla P - \nu_0 \nabla^2 \mathbf{v} - \tfrac{1}{3}\nu_0 \nabla(\nabla.\mathbf{v}) = \mathbf{f}, \tag{3.1 b}$$

$$\frac{\partial S}{\partial t} - \tfrac{4}{3}\nu_0 \nabla S - \tfrac{4}{3}(\gamma - 1)\nu_0 \nabla^2 P = \frac{\underset{\sim}{Q}}{\rho_0 C_p T_0}. \tag{3.1 c}$$

We can describe the kinematic field by two new independent variables, the vorticity $\mathbf{\Omega}$ and the specific dilatation rate q (the rate of volume increase per unit volume of the fluid) where

$$\mathbf{\Omega} = \nabla \times \mathbf{v}, \qquad q = \nabla.\mathbf{v}. \tag{3.2}$$

Taking the curl and divergence of (3.1 b), we have

$$\frac{\partial \mathbf{\Omega}}{\partial t} - \nu_0 \nabla^2 \mathbf{\Omega} = \nabla \times \mathbf{f}, \tag{3.1 b$'$}$$

$$\frac{\partial q}{\partial t} - \tfrac{4}{3}\nu_0 \nabla^2 q = -a_0^2 \nabla^2 P + \nabla.\mathbf{f}. \tag{3.1 b$''$}$$

The two equations (3.1 b)′ and (3.1 b)″ are equivalent to (3.1 b), since once $\mathbf{\Omega}$ and q are known, the velocity field can be calculated from (3.2) and the appropriate boundary conditions.

By a slight manipulation as indicated below, the system of equations (3.1 a), (3.1 b)′, (3.1 b)″ and (3.1 c) can be further combined into the following four equations:

$$\frac{\partial \mathbf{\Omega}}{\partial t} - \nu_0 \nabla^2 \mathbf{\Omega} = \nabla \times \mathbf{f}, \tag{3.3 a}$$

$$\frac{1}{a_0^2}\frac{\partial^2 P}{\partial t^2} - \nabla^2 P - \frac{4\gamma}{3}\frac{\nu_0}{a_0^2}\frac{\partial}{\partial t}(\nabla^2 P) = \frac{1}{a_0^2}\left[\left(\frac{\partial}{\partial t} - \tfrac{4}{3}\nu_0 \nabla^2\right)\frac{m}{\rho_0} - \nabla.\mathbf{f} + \frac{\partial}{\partial t}\left(\frac{\underset{\sim}{Q}}{\rho_0 C_p T_0}\right)\right], \tag{3.3 b}$$

$$\frac{\partial S}{\partial t} - \tfrac{4}{3}\nu_0 \nabla^2 S = \tfrac{4}{3}(\gamma - 1)\nu_0 \nabla^2 P + \frac{\underset{\sim}{Q}}{\rho_0 C_p T_0}, \tag{3.3 c}$$

$$q = -\frac{\partial P}{\partial t} + \frac{\partial S}{\partial t} + \frac{m}{\rho_0}. \tag{3.3 d}$$

Equation (3.3 a) is the same as (3.1 b); (3.3 c) is the same as (3.1 c); (3.3 d) is obtained by substituting (3.2) into (3.1 a); (3.3 b) is obtained by eliminating S and q from (3.1 b)″, (3.3 d) and (3.3 c).

The system of equations (3.2) and (3.3) can be conveniently split into three sub-systems as follows. Let us write

$$\mathbf{\Omega} = \mathbf{\Omega}_\Omega + \mathbf{\Omega}_p + \mathbf{\Omega}_s, \tag{3.4 a}$$

$$P = P_\Omega + P_p + P_s, \tag{3.4 b}$$

etc.; then

$$\frac{\partial \mathbf{\Omega}_\Omega}{\partial t} - \nu_0 \nabla^2 \mathbf{\Omega}_\Omega = \nabla \times \mathbf{f}, \tag{3.5 a}$$

$$P_\Omega = 0, \qquad S_\Omega = 0, \qquad q_\Omega = 0, \tag{3.5 b}$$

$$\nabla \times \mathbf{v}_\Omega = \mathbf{\Omega}_\Omega, \qquad \nabla . \mathbf{v}_\Omega = 0; \tag{3.5 c}$$

$$\mathbf{\Omega}_p = 0, \tag{3.6 a}$$

$$\frac{\partial^2 P_p}{\partial t^2} - a_0^2 \nabla^2 P_p - \tfrac{4}{3}\gamma \nu_0 \frac{\partial}{\partial t} \nabla^2 P_p = \left(\frac{\partial}{\partial t} - \tfrac{4}{3}\nu_0 \nabla^2\right)\frac{m}{\rho_0} - \nabla . \mathbf{f} + \frac{\partial}{\partial t}\frac{Q}{\rho_0 C_p T_0}, \tag{3.6 b}$$

$$\frac{\partial S_p}{\partial t} - \tfrac{4}{3}\nu_0 \nabla^2 S_p = \tfrac{4}{3}(\gamma - 1)\nu_0 \nabla^2 P_p, \tag{3.6 c}$$

$$q_p = -\frac{\partial P_p}{\partial t} + \frac{\partial S_p}{\partial t} + \frac{m}{\rho_0}, \tag{3.6 d}$$

$$\nabla \times \mathbf{v}_p = 0, \qquad \nabla . \mathbf{v}_p = q_p; \tag{3.6 e}$$

$$\mathbf{\Omega}_s = 0, \qquad P_s = 0, \tag{3.7 a}$$

$$\frac{\partial S_s}{\partial t} - \tfrac{4}{3}\nu_0 \nabla^2 S_s = \frac{Q}{\rho_0 C_p T_0}, \tag{3.7 b}$$

$$q_s = \frac{\partial S_s}{\partial t}, \tag{3.7 c}$$

$$\nabla \times \mathbf{v}_s = 0, \qquad \nabla . \mathbf{v}_s = q_s. \tag{3.7 d}$$

Of course, there are infinitely many ways in which we can split up the equations (3.2) and (3.4)*. However, the above choice is a convenient one since (3.5), (3.6) and (3.7) each represent individually familiar phenomena.

Thus (3.5) is identical with the equations describing the production, convection and dissipation of weak vorticity fluctuations in a viscous *incompressible* medium and will be called the *vorticity mode*. Such a flow field does not generate pressure fluctuations of the same order, since the latter is proportional to the square of the velocity fluctuation and hence is negligible when the vorticity fluctuation is weak. For a similar reason, there is no production of entropy fluctuation, since viscous dissipation is again a second-order quantity. Finally, the velocity field is solenoidal as it should be for an incompressible flow.

* It is possible to decompose the sound mode further into two types of physically distinct fluid motion: (1) that due to mass sources in an incompressible medium which is characterized by $P_q = 0$, $S_q = 0$, $\nabla . \mathbf{v}_q = q_q = m/\rho_0$, $\mathbf{\Omega}_q = \nabla \times \mathbf{v}_q = 0$; and (2) that associated with the propagation of pressure waves in a compressible medium which is characterized by (3.6 a), (3.6 b), (3.6 c), (3.6 e) and $q_p = -\partial P_p/\partial t + \partial S_p/\partial t$,

Equation (3.6) is identical with the equations describing the production, propagation and absorption of pressure waves in a viscous compressible heat-conductive medium, and will be called the *sound mode*. As a result of absorption, there will be entropy changes accompanying the pressure fluctuations. These entropy changes are denoted by S_p in (3.6). The velocity field is irrotational as is expected for a sound field.

Finally, (3.7) can be recognized as the equations describing the production, convection and diffusion of hot spots in a heat-conducting fluid medium and will be called the *entropy mode*. Such a diffusion phenomenon does not generate pressure fluctuations and produces only a weak irrotational velocity field, necessary for the conservation of mass while the density varies.

The vorticity mode is the type of fluid motion most frequently encountered in the mechanics of viscous incompressible fluid. The sound mode is the type of fluid motion discussed in acoustics and in the theory of compressible fluids. The entropy mode has been the main subject of investigation in the theory of heat transfer in fluids. We thus see that the solution of (3.1) can always be thought of as consisting of the superposition of three basic modes of fluctuations that are familiar in branches of fluid mechanics. Naturally, not all the three modes of fluctuations will enter significantly into a particular problem, for example, in turbulent heat transfer problems and high-speed shear flows, the fluctuations consist principally of the vorticity and entropy modes.

We now assume that the space-time domain G does not contain any solid boundaries (this is the second condition which limits the size of G). In such a case, each of the three modes of fluctuations evolves as though the others were absent. Should G contain some solid boundaries, the last conclusion is no longer true, since the boundary conditions are imposed upon the resultant flow variables, such as the velocity vector $\mathbf{v}$ or temperature T, and not separately upon the individual modes (see, for example, Trilling (1955)).

Decomposition of a small disturbance into different modes of fluctuations has been made previously by several authors, Carrier & Carlson (1946), Lagerstrom, Cole & Trilling (1949), Cole & Wu (1952), Wu (1952), Kovásznay (1953). The terminology and classification adopted here follows closely that of Kovásznay. The modes of fluctuations are re-defined here to include the influence of body forces, mass and heat sources. The fluid dynamic effects produced by these agents will be the subject of discussion in the next section.

When the space-time domain G is not sufficiently small the intensity of fluctuation is not small and terms of the order of second and higher powers of α in (2.12) must be retained. Since equation (2.16) has the same form for $n = 1$ as for $n > 1$, the flow field can still be thought of as consisting of the three basic modes of fluctuations. Since the apparent source terms $m^{(n)}$, $\mathbf{f}^{(n)}$, $Q^{(n)}$, in (2.16) are functions of the fluctuations $P^{(k)}$, $S^{(k)}$, $\mathbf{v}^{(k)}$ where $k < n$, the terms $P^{(n)}$, $S^{(n)}$, $\mathbf{v}^{(n)}$ may be interpreted as the pressure, entropy and velocity fluctuations generated by the nth-order interactions of the three modes of fluctuations. Of all these interactions the most important are the bilateral interactions of the three modes which contribute to the values of $P^{(2)}$, $S^{(2)}$ and $\mathbf{v}^{(2)}$.

The entire analysis is partly motivated by the interest in compressible flow turbulence and like phenomena. In those cases the fluctuation levels in the entire space-time domain are of the order of a few per cent of the corresponding mean values. The theory described here is valid for interactions of any order provided the process converges in G. Its application to the bilateral interactions of the various modes of fluctuations will be given in § 5 and § 6.

4. Effects of mass addition, body forces and heat sources

(1) *Effects of mass addition*

From equations (3.5), (3.6) and (3.7), we see that the effect of mass addition is to produce the *sound mode.* Closer examination of (3.6) reveals that the effects of mass addition are really two-fold. When the fluid is injected into the medium at a point or a region, it displaces the fluid that was originally there. Generally, the movement of the displaced fluid in turn generates pressure waves which propagate into the surrounding medium. However, mass addition with a spatio-temporal pattern obeying the diffusion equation does not cause pressure fluctuation, see (3.6 b). It induces an incompressible source flow, see (3.6 d), with a velocity potential ϕ_p obeying a Poisson-type equation

$$\nabla^2 \phi_p = \frac{m}{\rho_0}. \tag{4.1}$$

(2) *Effects of body force*

Referring to (3.5), (3.6) and (3.7), we conclude that the effects produced by body forces are two-fold. They generate the vorticity mode and the sound mode. If the force field is decomposed into two component fields, an irrotational and solenoidal field, the irrotational force component produces the sound mode (cf. (3.6 b)) and cannot produce the vorticity mode (cf. (3.5 a)). On the other hand, the solenoidal component produces the vorticity mode and cannot produce the sound mode. Consequently, a harmonic force field (which is both irrotational and solenoidal) can produce neither vorticity fluctuations nor pressure fluctuations. Its only effect is to produce a harmonic velocity field given by (see (3.1 b))

$$\frac{\partial \mathbf{v}_H}{\partial t} = \mathbf{f}_H. \tag{4.2}$$

(3) *Effects of heat addition*

From (3.5), (3.6) and (3.7), we see that addition of heat to a medium generates the entropy mode and the sound mode. Physically this is clear for heat addition to a gas produces two effects. Firstly, it increases the temperature of the gas (and hence the entropy) and, secondly, it causes an expansion of the heated fluid element. The first action is a more localized one and is described by the entropy mode. On the other hand, the expansion of the volume occupied by the heated fluid produces in the surrounding medium pressure waves, much like those generated by the inflation of a solid body. The action produced by the expansion of the heated fluid is thus described by the sound mode.

At the end of the preceding section, we showed that the fluctuations $P^{(n)}$, $S^{(n)}$, $Q^{(n)}$ can be interpreted as being generated by an apparent mass addition $m^{(n)}$, body force $\mathbf{f}^{(n)}$ and heat sources $Q^{(n)}$, resulting from the nth-order interaction of the various modes of fluctuations. The effects produced in the flow field by a particular type of interaction will then depend upon the manner in which the particular interaction affects the value of $m^{(n)}$, $\mathbf{f}^{(n)}$, $Q^{(n)}$. When the particular interaction gives rise predominantly to a contribution to $m^{(n)}$, we shall describe the effects produced as 'mass-like effects', and we shall speak of the interactions as having a 'mass-like action'. Likewise, we can speak of 'force-like effects', 'heat-like effects' or 'force-like actions' and 'heat-like actions'. Thus, for example, the bilateral vorticity–vorticity interaction will be shown to produce mainly a force-like effect, and a slight heat-like effect. A systematic study of the effects produced by bilateral interactions of the various modes of fluctuations is presented in the next section.

5. Effects of bilateral interactions of the various modes of fluctuations

General considerations

In § 3 we saw that the first-order system can be considered as the sum of three basic modes of fluctuations. Their contributions to the thermodynamic and kinematic variables are, from (3.4), (3.5), (3.6) and (3.7),

$$\mathbf{\Omega}^{(1)} = \mathbf{\Omega}^{(1)}_{\Omega}, \qquad P^{(1)} = P^{(1)}_{p}, \qquad S^{(1)} = S^{(1)}_{p} + S^{(1)}_{s}, \tag{5.1 a}$$

$$\mathbf{v}^{(1)} = \mathbf{v}^{(1)}_{\Omega} + \mathbf{v}^{(1)}_{p} + \mathbf{v}^{(1)}_{s}. \tag{5.1 b}$$

Substitution of (5.1) into (2.17) leads to the following expressions for the second-order source functions in terms of the separate first-order mode contributions as indicated by the double subscripts. (For example, $\mathbf{f}_{\Omega p}$ represents the apparent body force produced by the vorticity–sound interaction.)

$$m^{(2)} = \rho_0[m_{\Omega\Omega} + m_{pp} + m_{ss} + m_{\Omega p} + m_{ps} + m_{s\Omega}], \tag{5.2 a}$$

$$\mathbf{f}^{(2)} = \mathbf{f}_{\Omega\Omega} + \mathbf{f}_{pp} + \mathbf{f}_{ss} + \mathbf{f}_{\Omega p} + \mathbf{f}_{ps} + \mathbf{f}_{s\Omega}, \tag{5.2 b}$$

$$Q^{(2)} = \rho_0 C_p T_0[Q_{\Omega\Omega} + Q_{pp} + Q_{ss} + Q_{\Omega p} + Q_{ps} + Q_{s\Omega}]. \tag{5.2 c}$$

The 18 interaction terms can be computed from equations (2.17) and a brief examination leads us immediately to the conclusion that bilateral interactions of the three basic modes produce in general fluctuations in all three modes. In this case we have all 18 second-order interaction terms as indicated by Kovásznay (1953).

It is a natural question to ask whether all these 18 terms are equally important. Naturally such questions can be answered only if more is stated about initial conditions and boundary conditions. In a concrete flow field these are imposed by grids, coolers, fans, etc. One mode can often be neglected and this immediately reduces the number of interaction terms by one-half.

In the next section an order of magnitude estimate of the non-linear interactions will be carried out for an important special example still having sufficient generality to illustrate the nature of these second-order interactions.

6. Bilateral interactions of the three basic modes of fluctuations

Let us assume that the particular region of flow is far away from the boundaries and all three modes of fluctuations are present. Each mode of fluctuations can be pictured as a superposition of a large (or infinite) number of Fourier components. Typical components for each of the three modes of fluctuations have been discussed by Kovásznay (1953), or can be found directly from (3.5), (3.6) and (3.7) with m, $\mathbf{f}$ and Q put equal to zero. These are summarized below.

(1) *The vorticity mode*

The typical component of the vorticity fluctuation is given by

$$\mathbf{\Omega}^{(1)} = \mathbf{\Omega}_0^{(1)} \exp[i\mathbf{k}_\Omega . \mathbf{x} - \nu_0 k_\Omega^2 t], \tag{6.1 a}$$

where $\mathbf{\Omega}_0^{(1)} . \mathbf{k}_\Omega = 0$ and $\mathbf{\Omega}_0^{(1)}$ is the complex amplitude fluctuation. $\mathbf{k}_\Omega$ is the wave-number vector, t denotes time and $\mathbf{x}$ the position vector. The corresponding velocity fluctuation is

$$\mathbf{v}_\Omega^{(1)} = i\frac{\mathbf{k}_\Omega \times \mathbf{\Omega}_0^{(1)}}{k_\Omega^2} \exp[i\mathbf{k}_\Omega . \mathbf{x} - \nu_0 k_\Omega^2 t]. \tag{6.1 b}$$

(2) *The sound mode*

The typical component of the pressure fluctuation is given by

$$P^{(1)} = P_0^{(1)} \exp[i\mathbf{k}_p . \mathbf{x} - ct], \tag{6.2 a}$$

where $P_0^{(1)}$ is the complex amplitude; $\mathbf{k}_p$, the wave-number vector, and c is the complex number

$$c = a_0 k_p \left(\frac{2\gamma}{3}\frac{\nu_0 k_p}{a_0} - i\left\{1 - \frac{4\gamma^2}{9}\left(\frac{\nu_0 k_p}{a}\right)^2\right\}^{1/2}\right). \tag{6.2 b}$$

The real part of c gives the rate of damping while the imaginary part gives the frequency of oscillation. Note that the absolute magnitude of c is of the order of $a_0 k_p$, since in general $(\nu_0 k_p/a_0) \ll 1$. (See also the paragraph following equation (6.4).) The entropy generated as a result of the dissipation of the pressure waves is given by

$$S_\nu^{(1)} = \frac{\frac{4}{3}(\gamma - 1)\nu_0 k_p^2}{c - \frac{4}{3}\nu_0 k_p} P_0^{(1)} \exp[i\mathbf{k}_p . \mathbf{x} - ct]. \tag{6.2 c}$$

The velocity fluctuation associated with the sound mode is

$$\mathbf{v}_p^{(1)} = \frac{ia_0^2 k_p}{c - \frac{4}{3}\nu_0 k_p} P_0^{(1)} \exp[i\mathbf{k}_p \,.\, \mathbf{x} - ct]. \tag{6.2 d}$$

(3) *The entropy mode*

The typical component of the entropy fluctuation associated with the entropy mode is

$$S_s^{(1)} = S_{s0}^{(1)} \exp[i\mathbf{k}_s \,.\, \mathbf{x} - \tfrac{4}{3}\nu_0 k_s^2 t], \tag{6.3 a}$$

where $S_{s0}^{(1)}$ is the complex amplitude and $\mathbf{k}_s$ is the wave-number vector. The corresponding velocity fluctuation is

$$\mathbf{v}_s^{(1)} = i\tfrac{4}{3}\nu_0 \mathbf{k}_s S_{s0}^{(1)} \exp[i\mathbf{k}_s \,.\, \mathbf{x} - \tfrac{4}{3}\nu_0 k_s^2 t]. \tag{6.3 b}$$

(4) *Relative orders of magnitude*

We are now ready to estimate the relative importance of the various interaction terms. By inspecting (5.3), (5.4) and (5.5) from the point of view of substitution of periodic solutions given by (6.1), (6.2) and (6.3), we can assert the following facts:

(*a*) Fluctuations in each mode may be characterized by two quantities, namely, an amplitude and a length scale.

(*b*) The medium has an intrinsic length scale, namely ν_0/a_0.

(*c*) In the interaction formulae, differentiation with respect to space coordinates does occur but integration does not; consequently, the terms containing derivatives decrease monotonically with decreasing wave-numbers (with increasing length scales).

We shall discuss the more complicated case where all three fluctuation modes have amplitude and scales, respectively, of the same order of magnitude. If one of the amplitudes or one of the wave-numbers is smaller than the other two, this can only simplify matters and never complicate them.

Whenever a quantity is differentiated with respect to the space coordinates, factors of k and k^2, etc., will appear. We shall assume that all three characteristic wave-numbers are of the same order of magnitude, i.e. k_Ω/k, k_p/k, k_s/k are each of order one, where k is the largest among k_Ω, k_p and k_s.

The amplitudes of the fluctuations have been characterized by the non-dimensional parameter α. The length scale of the fluctuations will be characterized by another non-dimensional parameter ϵ, defined as the ratio of the intrinsic scale of the medium ν_0/a_0 to the length scale of the disturbances k, that is,

$$\epsilon = \nu_0 k / a_0. \tag{6.4}$$

Clearly ϵ can be regarded as the Knudsen number of the disturbances, when it is small the inertial effects will dominate over heat conduction and viscous effects. The magnitude of ϵ can be estimated for atmospheric conditions for which $\nu_0 = 0{\cdot}15\,\text{cm}^2\,\text{sec}^{-1}$ and $a_0 = 30\,000\,\text{cm}\,\text{sec}^{-1}$. The intrinsic length (which is simply proportional to the mean free path of the gas) is $\nu_0/a_0 = 0{\cdot}5 \times 10^{-5}$ cm. Even in the extreme case when the characteristic wavelength of the disturbance is as small as 1 mm, $\epsilon = 3{\cdot}14 \times 10^{-4}$.

The typical bilateral interaction term is of the order of α^2 and we shall consistently neglect terms of order α^3 and higher. If, however, $\epsilon \ll 1$ as is usually the case and especially if $\epsilon < \alpha$ we shall find bilateral interaction terms that are of the order $\alpha^2\epsilon$. (These usually represent higher derivatives associated with viscous or heat conduction terms.) We propose to neglect these terms as they will be smaller than the third-order interaction terms. Furthermore, if $\epsilon \ll \alpha$ we may even neglect the viscous and heat conduction terms in the first-order problem and obtain the simplest set of equations

$$\frac{\partial \mathbf{\Omega}^{(0)}}{\partial t} = 0, \tag{6.5 a}$$

$$\frac{\partial S^{(0)}}{\partial t} = 0, \tag{6.5 b}$$

$$\frac{\partial^2 P^{(0)}}{\partial t^2} - \alpha_0^2 \nabla^2 P^{(0)} = 0. \tag{6.5 c}$$

The superscript (0) emphasizes the difference between these equations and (3.5 a), (3.6 b) and (3.7 b).

This set of equations was presented by Kovásznay (1953) and represents a consistent 'zero order approximation' for weak fluctuation fields when the time interval of observation is short. The space domain of validity may or may not be also small depending on whether or not α increases with increasing space domain. (In homogeneous turbulence α may approach a limit for increasing domain.)

Equation (6.5 a) represents a 'frozen pattern' of vorticity (or solenoidal velocity) and when it is applied to homogeneous turbulence in a wind tunnel it predicts an unchanged flow pattern carried by the mean flow velocity as suggested by Taylor's hypothesis. Experimental verification of the frozen pattern concept by space-time correlation has been carried out in low speed flow for grid turbulence (Favre *et al.* 1954) and also the outer portion of the turbulent boundary layer (Favre *et al.* 1957).

Equation (6.5 b) indicates a 'frozen pattern' behaviour for temperature spots and was similarly used for evaluating hot-wire measurements in low-speed turbulent temperature fields. Equation (6.5 c) represents the well-known equations of sound propagation of the pressure field.

Equations (6.5) represent the most modest approximation for predicting the temporal behaviour of the three modes. Their value in interpreting hot-wire measurements in a high-speed flow is equal to that of Taylor's hypothesis as used for interpreting hot-wire measurements in low-speed turbulent flows.

We shall consider the amplitudes of the three modes to have orders of magnitude when the velocity fluctuations in the sound mode are comparable to the velocity fluctuations in the vorticity mode and when the temperature fluctuations in the sound mode are comparable to the temperature fluctuations in the entropy mode, i.e. $|v_p^{(1)}| \approx |v_\Omega^{(1)}|$, $|T_p^{(1)}| \approx |T_s^{(1)}|$.

Space and time derivatives can be estimated from (6.1), (6.2) and (6.3) as follows. By choosing the reference space scale as k^{-1} and the reference time scale as $(a_0 k)^{-1}$ all the non-dimensional space derivatives will have the same order of magnitude as the quantity which is differentiated (except for factors k_Ω/k, k_p/k, k_s/k that were assumed earlier to be of order unity).

The orders of magnitude of the non-dimensional time derivatives differ depending upon the 'hyperbolic' (sound) or 'parabolic' (vorticity and entropy) character of the mode. For the sound mode the non-dimensional time derivative is of the same order as the space derivative (since the speed of sound was used as the reference velocity for forming non-dimensional quantities). For the vorticity and entropy modes, however, the non-dimensional time derivatives have order of magnitude $(k_\Omega/k)^2\epsilon$ or $(k_s/k)^2\epsilon$ times the quantities that differentiated. Indeed, differentiation of (6.1) with respect to t produces a factor $\nu_0 k_\Omega^2$ and non-dimensionalization of t by $(a_0 k)^{-1}$ results in the factor $(\nu_0 k_\Omega^2)/(a_0 k) = (k_\Omega/k)^2\epsilon$. Physically, this is clear for the rate of decay of the parabolic modes is an extremely slow process when their length scale is very large compared to the mean free path and when time is measured in units of the period of sound oscillation.

The above estimate for the order of magnitudes* of the various quantities associated with the sound mode can be summarized as follows:

$$\left.\begin{aligned} &P^{(1)}, \frac{v_p^{(1)}}{a_0}, \frac{T_p^{(1)}}{T_0} = O(\alpha), \qquad S_p^{(1)} = O\left(\frac{k_p}{k}\alpha\epsilon\right), \\ &\frac{\partial P^{(1)}}{\partial x_i}, \frac{\partial P^{(1)}}{\partial t}, \frac{1}{a_0}\frac{\partial \mathbf{v}_p^{(1)}}{\partial x_i}, \frac{1}{a_0}\frac{\partial v_p^{(1)}}{\partial t}, \frac{1}{T_0}\frac{\partial T_p^{(1)}}{\partial x_i}, \quad \frac{1}{T_0}\frac{\partial T_p^{(1)}}{\partial t} = O\left(\frac{k_p}{k^2}\alpha\right), \\ &\frac{\partial S_p^{(1)}}{\partial x_i}, \quad \frac{\partial S_p^{(1)}}{\partial t} = O\left(\frac{k_p^2}{k^2}\alpha\epsilon\right). \end{aligned}\right\} \tag{6.6}$$

According to our basic assumption in this section, the intensity of the vorticity mode must be such that $v_\Omega^{(1)}/a_0 = O(v_p^{(1)}/a_0) = O(\alpha)$. The order of the vorticity properly non-dimensionalized is then $\Omega^{(1)}/a_0 k_\Omega = O(\alpha)$.

The estimates concerning the vorticity mode can also be summarized simply as follows:

$$\left.\begin{aligned} &\frac{\Omega^{(1)}}{a_0 k_\Omega} = O\left(\frac{k_\Omega}{k}\alpha\right), && \frac{v_\Omega^{(1)}}{a_0} = O(\alpha), \\ &\frac{1}{a_0 k}\frac{\partial \mathbf{\Omega}_\Omega^{(1)}}{\partial x_i} = O\left(\frac{k_\Omega^2}{k^2}\alpha\right), && \frac{1}{a_0}\frac{\partial \mathbf{v}_\Omega^{(1)}}{\partial x_i} = O\left(\frac{k_\Omega}{k}\alpha\right), \\ &\frac{1}{a_0 k}\frac{\partial \mathbf{\Omega}_\Omega^{(1)}}{\partial t} = O\left(\frac{k_\Omega^3}{k^3}\alpha\epsilon\right), && \frac{1}{a_0}\frac{\partial \mathbf{v}_\Omega^{(1)}}{\partial t} = O\left(\frac{k_\Omega^2}{k^2}\alpha\epsilon\right). \end{aligned}\right\} \tag{6.7}$$

* The terms k_p/k, etc., are retained to facilitate arguments in the sub-cases where one of the wave-numbers k_p, k_Ω and k_s is small.

Finally, in accordance with the basic assumption, the temperature fluctuation associated with the entropy mode must be $T_s^{(1)}/T_0 = O(T_p^{(1)}/T_0) = O(\alpha)$. Now from (2.18) and (3.7 a), $T_s^{(1)}/T_0 = S_s^{(1)}$. It follows that $S_s^{(1)} = O(\alpha)$, and from (6.3 b), we find

$$\frac{v_s^{(1)}}{a_0} = O\left(\frac{k_s}{k}\alpha\epsilon\right).$$

Consequently, the velocity associated with the entropy mode is usually a very small quantity. This is also clear from (3.7 c) and (3.7 d). Evidently this velocity is induced as a result of the diffusion of entropy spots, which is a rather slow process when time is measured in units of the period of sound oscillation of a comparable spatial scale.

The estimates for the order of magnitude of the various quantities associated with the entropy mode are now summarized*:

$$\left.\begin{aligned} &S_s^{(1)}, \frac{T_s^{(1)}}{T_0} = O(\alpha), && \frac{v_s^{(1)}}{a_0} = O\left(\frac{k_s}{k}\alpha\epsilon\right), \\ &\frac{\partial S_s^{(1)}}{\partial x_i}, \frac{1}{T_0}\frac{\partial T_s^{(1)}}{\partial x_i} = O\left(\frac{k_s}{k}\alpha\epsilon\right), && \frac{1}{a_0}\frac{\partial \mathbf{v}_s^{(1)}}{\partial x_i} = O\left(\frac{k_s^2}{k^2}\epsilon\alpha\right), \\ &\frac{\partial S_s^{(1)}}{\partial t}, \frac{1}{T_0}\frac{\partial T_s^{(1)}}{\partial t} = O\left(\frac{k_s^2}{k^2}\alpha\epsilon\right), && \frac{1}{a_0}\frac{\partial \mathbf{v}_s^{(1)}}{\partial t} = O\left(\frac{k_s^3}{k^3}\epsilon^2\alpha\right). \end{aligned}\right\} \tag{6.8}$$

By substituting the mode contributions in equations (2.17) the significant bilateral source terms can be readily obtained. Since only bilateral interactions of the first-order solutions are computed here we shall drop the superscript (1) in all subsequent equations. The bilateral source terms are

$$m_{\Omega\Omega} = 0, \tag{6.9 a}$$

$$m_{pp} = -\nabla.(P_p\mathbf{v}_p) + \frac{\gamma-1}{2}\frac{\partial}{\partial t}P_p^2 + O(\epsilon\alpha^2), \tag{6.9 b}$$

$$m_{ss} = O(\epsilon\alpha^2), \tag{6.9 c}$$

$$m_{\Omega p} = -\nabla.(P_p\mathbf{v}_\Omega) + O(\epsilon\alpha^2), \tag{6.9 d}$$

$$m_{ps} = \nabla.(S_s\mathbf{v}_p) + S_s\frac{\partial P_p}{\partial t} + O(\epsilon\alpha^2), \tag{6.9 e}$$

$$m_{s\Omega} = \nabla.(S_s\mathbf{v}_\Omega) + O(\epsilon\alpha^2); \tag{6.9 f}$$

$$\mathbf{f}_{\Omega\Omega} = -(\mathbf{v}_\Omega.\nabla)\mathbf{v}_\Omega, \tag{6.10 a}$$

$$\mathbf{f}_{pp} = -P_p\frac{\partial \mathbf{v}_p}{\partial t} - \tfrac{1}{2}\nabla(v_p^2) + O(\epsilon\alpha^2), \tag{6.10 b}$$

$$\mathbf{f}_{ss} = O(\epsilon\alpha^2), \tag{6.10 c}$$

$$\mathbf{f}_{p\Omega} = -[\nabla(\mathbf{v}_p.\mathbf{v}_\Omega) - \mathbf{v}_p\times\mathbf{\Omega}_\Omega] + O(\epsilon\alpha^2), \tag{6.10 d}$$

$$\mathbf{f}_{ps} = S_s\frac{\partial \mathbf{v}_p}{\partial t} + O(\epsilon\alpha^2), \tag{6.10 e}$$

$$\mathbf{f}_{s\Omega} = O(\epsilon\alpha^2). \tag{6.10 f}$$

* Here we assume that the quantity $\mu_0' T_0$ is of the same order of magnitude as μ_0. This is indeed the case for air if Sutherland's formula for viscosity is used, $(\mu_0'/\mu_0)T_0 = 1{\cdot}5 - T_0/(T_0+C)$ where $\mu_0' = (d\mu/dT)_{T=T_0}$ and $C = 140°$ K.

Finally, (5.5) can be simplified to

$$Q_{\Omega\Omega}, Q_{pp}, Q_{ss}, Q_{\Omega p} = O(\epsilon\alpha^2), \tag{6.11 a}$$

$$Q_{ps} = -(\mathbf{v}_p . \nabla) S_s + O(\epsilon\alpha^2), \tag{6.11 b}$$

$$Q_{s\Omega} = -(\mathbf{v}_\Omega . \nabla) S_s + O(\epsilon\alpha^2). \tag{6.11 c}$$

The various types of interactions can now be described in greater detail.

Vorticity–vorticity interactions

According to (6.9 a), (6.10 a) and (6.11 a) the bilateral vorticity–vorticity interaction produces mainly a force-like effect. In addition, there is a slight heat-like effect (of order ϵ smaller). The apparent body force is $-(\mathbf{v}_\Omega . \nabla)\mathbf{v}_\Omega$ or, in tensor notation,

$$[f_{\Omega\Omega}]_i = -v_{\Omega_j} \frac{\partial}{\partial x_j} v_{\Omega i} = -\frac{\partial}{\partial x_j}[v_{\Omega i} v_{\Omega j}].$$

Since this force field is in general neither solenoidal nor irrotational, it generates both the vorticity mode and sound mode. The production of the vorticity fluctuation is of fundamental importance in the mechanics of turbulent motion. The production of sound by vorticity–vorticity interaction has been analysed in a general fashion by Lighthill (1952). Our result with regard to sound generation could have been anticipated from Lighthill's theory. However, speaking in terms of body force instead of stresses, this analysis brings out more clearly the fact that the solenoidal component of the force field plays no part in sound generation but is of basic importance in the energy transfer between different size eddies in the turbulent fluctuation field.

Sound–sound interaction

According to (6.9 b), (6.10 b) and (6.11 a), bilateral interactions of sound modes produce primarily the mass-like and force-like effects. In addition there is a slight heat-like effect indicating that only weak entropy fluctuations are generated by sound–sound interactions. A typical example of sound–sound interaction is the 'steepening' of the finite amplitude compressive waves and it is well known that the deviation from the isentropic change through even a moderately weak shock wave is proportional only to the third power of the shock strength. This particular conclusion also suggests that the sound–sound interaction may lead to only a slight production of the vorticity fluctuations. From (6.10 b), we have

$$\mathbf{f}_{pp} = -P\frac{\partial \mathbf{v}_p}{\partial t} - \tfrac{1}{2}\nabla v_p^2 + O(\epsilon\alpha^2).$$

Furthermore, the velocity induced by pressure waves propagating in a viscous compressible medium satisfies the equation

$$\frac{\partial \mathbf{v}_p}{\partial t} = -a_0^2 \nabla P + {}_3\nu_0 \nabla^2 \mathbf{v}_p,$$

so that $\mathbf{f}_{pp}$ can be rewritten as

$$\mathbf{f}_{pp} = -\tfrac{1}{2}\nabla[a_0^2 P^2 + v_p^2] + O(\epsilon\alpha^2).$$

Consequently, $\mathbf{f}_{pp}$ is principally an irrotational vector field, so that sound–sound interaction does produce only a slight vorticity fluctuation. There is, therefore, little possibility of altering the intensity of turbulent fluctuation in high speed flow through the random interaction of the pressure waves and, possibly, weak shock waves. We thus see that the main effect of sound–sound interaction is the production of the sound mode itself.

The net strength of the sound source, according to (3.3 b), (6.9 b), (6.10 b) and (6.11 a), is

$$\left(\frac{\partial}{\partial t} - \tfrac{4}{3}\nu_0 \nabla^2\right) m_{pp} - \nabla . \mathbf{f}_{pp} + \frac{\partial Q_{pp}}{\partial t} = \nabla . [\mathbf{v}_p(\nabla . \mathbf{v}_p) + (\mathbf{v}_p . \nabla)\mathbf{v}_p + a_0^2 \nabla(P_p^2)] + + \frac{\gamma-1}{2}\frac{\partial^2}{\partial t^2} P_p^2 + O(\epsilon\alpha^2),$$

where use has been made of (3.6 d) and (3.6 e) (with $m = 0$). In tensor notation, the last result can be written simply as

$$\frac{\partial^2}{\partial x_i \partial x_j}(v_{pi} v_{pj} + a_0^2 P_p^2 \delta_{ij}) + \frac{\gamma-1}{2}\frac{\partial^2}{\partial t^2} P_p^2 + O(\epsilon\alpha^2),$$

or alternatively

$$\frac{\partial^2}{\partial x_i \partial x_j}\left[v_{pi} v_{pj} + \frac{\gamma+1}{2} a_0^2 P_p^2 \delta_{ij}\right] + (\gamma-1)\left[\left(\frac{\partial P_p}{\partial t}\right)^2 - a_0^2(\nabla P_p)^2\right] + O(\epsilon\alpha^2).$$

Assuming that the flow fluctuations at infinity are zero, the distribution of source strength, when integrated over the entire flow field, in general yields a resultant strength different from zero indicating that there is a net generation besides self-scattering. (A discussion of self-scattering was given by Westervelt (1957).)

Entropy–entropy interaction

Our analysis clearly shows that the entropy–entropy interaction, though a theoretical possibility, is practically insignificant. From (6.9 c), (6.10 c) and (6.11 a), it is seen that it produces only a slight mass-like and a slight heat-like effect (both of order of $\epsilon\alpha^2$), and an even smaller force-like effect (of order $\epsilon^2\alpha^2$).

Vorticity–sound interaction

According to (6.9 d), (6.10 d) and (6.11 a), the vorticity–sound interaction produces a mass-like and force-like effect. It has only a slight heat-like effect and hence produces only very slight entropy fluctuations (i.e. of order of $\epsilon\alpha^2$).

Since the force field in general is not an irrotational field, there will be vorticity production as a result of interaction. The source strength in the production of vorticity fluctuations is $\nabla \times (\mathbf{v}_p \times \mathbf{\Omega}_\Omega)$. This term includes the effect of both the stretching and expansion of vortex tubes and the convection of vorticity under the action of sound fluctuation. In the case when a single shock wave is interacting with the turbulent flow, the theoretical calculation of Ribner (1954), which was explored experimentally by Kovásznay (1955), indicates that the effect is not very large, at least in that particular problem.

The vorticity–sound interaction also produces the sound mode. The resultant strength of the sound source as given by (3.7) can be simplified to

$$\left(\frac{\partial}{\partial t} - \tfrac{4}{3}\nu_0\nabla^2\right)m_{\Omega p} - \nabla.\mathbf{f}_{\Omega p} + \frac{\partial Q_{\Omega p}}{\partial t} = 2\nabla.[(\mathbf{v}_\Omega.\nabla)\mathbf{v}_p] + O(\epsilon\alpha^2).$$

In the last step of the derivation, use has been made of the relation

$$\frac{\partial P_p}{\partial t} = -\nabla.\mathbf{v}_p + O(\epsilon\alpha).$$

In tensor notation, the resultant source strength of the sound mode is simply $2\,\partial^2(\mathbf{v}_{\Omega i}\mathbf{v}_{pj})/\partial x_i\partial x_j$, a result which could have been anticipated from Lighthill's quadrupole radiation theory. Assuming that the flow fluctuations die down fast enough at infinity, integration of the source strength over the entire flow field yields no net contributions. Consequently, the sound mode produced by vorticity–sound interaction can be interpreted as the scattering of the incident sound wave by the vorticity fluctuations. Scattering of a shock wave by turbulent fluctuations has also been calculated by Ribner (1954).

Sound–entropy interaction

By (6.9 e), (6.10 e) and (6.11 b), the interaction of the sound and entropy modes produce all three basic modes of fluctuations.

Let us first examine the vorticity mode produced by the interaction. According to (3.3 a) and (6.10 e), the source term which produces vorticity fluctuation is

$$\nabla\times\left[S_s\frac{\partial\mathbf{v}_p}{\partial t}\right] + O(\epsilon\alpha^2) = \nabla S_s\times\frac{\partial\mathbf{v}_p}{\partial t} + O(\epsilon\alpha^2) = -a_0^2\nabla S_s\times\nabla P_p.$$

This is a familiar term in the theory of rotational flow. Physically, the vorticity produced by it can be interpreted as the consequence of the fact that the resultant force acting on any lump of fluid fails to pass through the centre of mass, and thus produces a torque and an angular acceleration. This result suggests that the action of pressure field on a stream carrying randomly distributed temperature spots in a turbulent stream may generate turbulent fluctuations and can modify the original turbulent fluctuations in the stream. In the only known experimental realization of this situation (spottily heated air accelerating through the nozzle of a supersonic wind-tunnel) the effect appeared unimportant (see Morkovin 1956, p. 8).

Next consider the sound mode generated by the interaction. The net sound source can be calculated from (3.6 b) and can readily be simplified to

$$\left(\frac{\partial}{\partial t} - \tfrac{4}{3}\mathbf{v}_0\nabla^2\right)m_{ps} - \nabla.\mathbf{f}_{ps} + \frac{\partial Q_{ps}}{\partial t} = -\nabla.\left[\frac{\partial}{\partial t}(s_s\mathbf{v}_p)\right] + O(\epsilon\alpha).$$

If the disturbance dies down at infinity fast enough, the resultant source strength integrated over the entire flow field is zero, so that the phenomenon can again be interpreted as the scattering of the incident sound wave by the entropy spots.

Finally, consider the generation of the entropy spottiness by the interaction. It follows from (3.7 b) and (6.11 b) that the source of entropy fluctuation is given by the term $-(\mathbf{v}_p \, . \, \nabla)S_s$. It can be regarded as a convection of heat by the motion of the sound field.

Special cases of generation of all three fluctuation modes in the interaction between an entropy disturbance and an infinite or wedge-attached shock wave were discussed theoretically by Chang (1955).

Entropy–vorticity interaction

According to (6.9 f), (6.10f) and (6.11 c), the entropy–vorticity interaction produces a mass-like and a heat-like effect as well as a slight force-like effect. Consequently, this type of interaction can only produce a very weak (of

	Sound source	Vorticity source	Entropy source
Sound–sound	' steepening ' and ' self-scattering ' $\frac{\partial^2(v_{pi}\, v_{pj})}{\partial x_i\, \partial x_j} + a_0^2\, \nabla^2(P_p^2) + \frac{\gamma-1}{2}\, \frac{\partial}{\partial t^2}\, P_p^2$	$O(\alpha^2 \epsilon)$	$O(\alpha^2 \epsilon)$
Vorticity–vorticity	' generation ' $\frac{\partial^2(v_{\Omega i}\, v_{\Omega j})}{\partial x_i\, \partial x_j}$	' self-convection ' $-v_{\Omega j}\, \frac{\partial \Omega_i}{\partial x_j} + \Omega_j\, \frac{\partial v_{\Omega i}}{\partial x_j}$	$O(\alpha^2 \epsilon)$
Entropy–entropy	$O(\alpha^2 \epsilon)$	$O(\alpha^2 \epsilon^2)$	$O(\alpha^2 \epsilon)$
Sound–vorticity	' scattering ' $2\, \frac{\partial^2(v_{\Omega i}\, v_{pj})}{\partial x_i\, \partial x_j}$	' vorticity convection ' $-v_{pj}\, \frac{\partial \Omega_i}{\partial x_j} + \Omega_j\, \frac{\partial v_{pi}}{\partial x_j} - \Omega_i\, \frac{\partial v_{pj}}{\partial x_j}$	$O(\alpha^2 \epsilon)$
Sound–entropy	' scattering ' $\frac{\partial^2}{\partial t \partial x_i}\, (S_s\, v_{pi})$	' generation ' $-\alpha_0^2(\nabla S_s) \times (\nabla P_p)$	' heat convection ' $-v_{pi}\, \frac{\partial S_s}{\partial x_i}$
Vorticity–entropy	$O(\alpha^2 \epsilon)$	$O(\alpha^2 \epsilon)$	' heat convection ' $-v_{\Omega i}\, \frac{\partial S_s}{\partial x_i}$

Table 1. Second-order non-linear interaction between modes of order α.

order $\epsilon\alpha^2$) vorticity fluctuation. The sound field produced turns out to be weak also because the expansion generated by the apparent mass addition is exactly balanced by the contraction following heat subtraction. This can best be seen if we calculate the net source strength for the sound mode produced. From (6.9 f) and (6.11 c), one concludes that

$$\left(\frac{\partial}{\partial t} - \tfrac{4}{3}\nu_0\nabla^2\right)m_{s\Omega} - \nabla.\mathbf{f}_{s\Omega} + \frac{\partial Q_{s\Omega}}{\partial t} = \frac{\partial}{\partial t}\nabla.[\mathbf{v}_\Omega S_s] - \frac{\partial}{\partial t}(\mathbf{v}_\Omega.\nabla)S_s + O(\epsilon\alpha^2)$$
$$= O(\epsilon\alpha^2),$$

since $\nabla.\mathbf{v}_\Omega = 0$. Consequently, the main effects of entropy–vorticity interaction are in the production of the entropy mode. This mode is generated as a result of the convection of the hot spots by velocity fluctuation associated with the vorticity mode. This type of interaction has a leading role in turbulent heat transfer phenomena, see Corrsin (1952).

In order to summarize all second-order effects table 1 was prepared. It gives the source strength for all three modes up to order α^2. Terms of order $\alpha^2\epsilon$ and $\alpha^2\epsilon^2$ are only indicated. Table 1 includes also a more or less intuitive label for each bilateral interaction, like 'generation' or 'scattering'. These are not intended as accurate descriptions but are merely to suggest the corresponding physical processes.

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